Time domain approach for acoustic liner analysis

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Presentation Outline

- **Procedures for implementing the Ingard-Myers impedance condition in time domain**
- **Instability issue of the Ingard-Myers Condition**
- **Proposed stabilization of the Ingard-Myers condition**
- **Numerical examples**

TD-FAST: Time Domain Acoustic Scattering Toolkit

Solving convective wave equation

$$\left(\frac{\partial}{\partial t} + \boldsymbol{U} \cdot \nabla\right)^2 p - c^2 \nabla^2 p = s(\boldsymbol{r}, t)$$

by Time Domain Boundary Integral Equation (TDBIE) formulation:

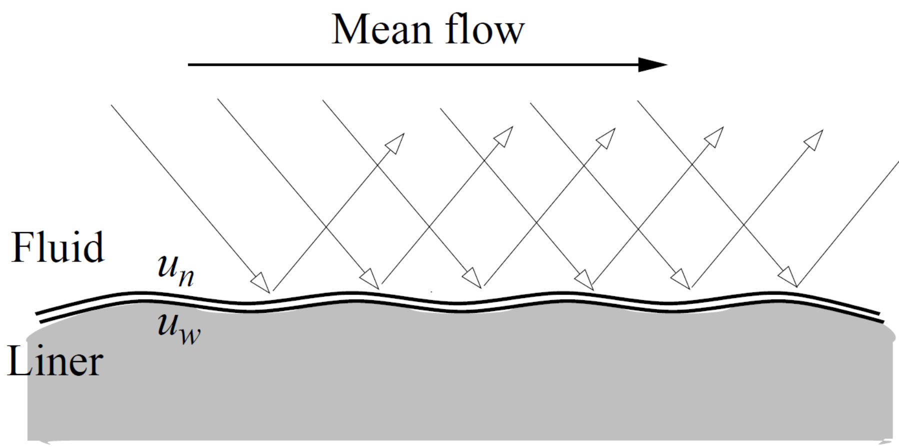
$$2\pi p(\boldsymbol{r}'_s, t') - \int_S \left[G_0 \frac{\partial p}{\partial \tilde{n}}(\boldsymbol{r}_s, t'_R) - \frac{\partial G_0}{\partial \bar{n}} \left(p(\boldsymbol{r}_s, t'_R) + \frac{\bar{R}}{c\alpha^2} \frac{\partial p}{\partial t}(\boldsymbol{r}_s, t'_R) \right) \right] d\boldsymbol{r}_s = Q(\boldsymbol{r}'_s, t')$$

On solid surfaces: $p_n = \dfrac{\partial p}{\partial \tilde{n}} = 0$

On lined surfaces:

p and p_n are coupled through the impedance condition

Ingard-Myers Impedance Boundary Condition



$$\frac{\hat{p}(\boldsymbol{r},\omega)}{\hat{u}_w(\boldsymbol{r},\omega)} = Z(\omega)$$

A thin vortex sheet assumed: $\zeta = \zeta(\boldsymbol{r},t)$, $u_w = \dfrac{\partial \zeta}{\partial t}$, $u_n = \dfrac{\partial \zeta}{\partial t} + \boldsymbol{U}\cdot\nabla\zeta$

Relation between u_w and u_n: $\dfrac{\partial u_n}{\partial t} = \dfrac{\partial u_w}{\partial t} + \boldsymbol{U}\cdot\nabla u_w$

Relation between u_n and $\dfrac{\partial p}{\partial n}$: $\dfrac{\partial u_n}{\partial t} + \boldsymbol{U}\cdot\nabla u_n + \dfrac{1}{\rho_0}\dfrac{\partial p}{\partial n} = 0$

[1] U. Ingard, “Influence of fluid motion past a plane boundary on sound reflection, absorption, and transmission,” J. Acoust. Soc. Am., Vol. 31, 1959.
[2] M. Myers, “On the acoustic boundary condition in the presence of flow,” J. Sound Vib. Vol. 71, 1980.

Time domain impedance condition between p and u_w

Assume a broadband impedance function of the following form:

$$Z(\omega) = (-i\omega)h_0 + R_0 + \sum_{m=1}^{M} \frac{A_m}{\lambda_m - i\omega} + \frac{1}{2}\sum_{\ell=1}^{L}\left[\frac{B_\ell + iC_\ell}{\alpha_\ell + i\beta_\ell - i\omega} + \frac{B_\ell - iC_\ell}{\alpha_\ell - i\beta_\ell - i\omega}\right]$$

Impedance condition $\dfrac{\hat{p}(\boldsymbol{r},\omega)}{\hat{u}_w(\boldsymbol{r},\omega)} = Z(\omega)$ converted to time domain:

$$p(\boldsymbol{r},t) = h_0\frac{\partial u_w}{\partial t} + R_0 u_w(\boldsymbol{r},t) + \sum_{m=1}^{M} A_m p_m^{(0)}(\boldsymbol{r},t) + \sum_{\ell=1}^{L}\left[B_\ell p_\ell^{(1)}(\boldsymbol{r},t) + C_\ell p_\ell^{(2)}(\boldsymbol{r},t)\right]$$

$$\frac{dp_m^{(0)}}{dt} + \lambda_m p_m^{(0)}(\boldsymbol{r},t) = u_w(\boldsymbol{r},t), \quad m = 1,\dots,M$$

$$\frac{dp_\ell^{(1)}}{dt} + \alpha_\ell p_\ell^{(1)}(\boldsymbol{r},t) + \beta_\ell p_\ell^{(2)}(\boldsymbol{r},t) = u_w(\boldsymbol{r},t), \quad \frac{dp_\ell^{(2)}}{dt} + \alpha_\ell p_\ell^{(2)}(\boldsymbol{r},t) - \beta_\ell p_\ell^{(1)}(\boldsymbol{r},t) = 0, \quad \ell = 1,\dots,L$$

Time domain impedance condition between p and u_n

By vortex sheet model: $\dfrac{\partial u_n}{\partial t} = \dfrac{\partial u_w}{\partial t} + \boldsymbol{U} \cdot \nabla u_w$

Impedance condition:

$$\frac{\partial p}{\partial t} + \boldsymbol{U} \cdot \nabla p(\boldsymbol{r}, t) = h_0 \frac{\partial^2 u_n}{\partial t^2} + R_0 \frac{\partial u_n}{\partial t} + \sum_{m=1}^{M} A_m p_m^{(0)}(\boldsymbol{r}, t) + \sum_{\ell=1}^{L} \left[B_\ell p_\ell^{(1)}(\boldsymbol{r}, t) + C_\ell p_\ell^{(2)}(\boldsymbol{r}, t) \right]$$

$$\frac{dp_m^{(0)}}{dt} + \lambda_m p_m^{(0)}(\boldsymbol{r}, t) = \frac{\partial u_n}{\partial t}, \quad m = 1, ..., M$$

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Time domain impedance condition between p and p_n

By Euler equation: $\frac{\partial u_n}{\partial t} + \boldsymbol{U} \cdot \nabla u_n + \frac{1}{\rho_0} \frac{\partial p}{\partial n} = 0$

We get the following Time Domain Impedance Boundary Condition:

$$\frac{\partial p}{\partial t} + 2\boldsymbol{U} \cdot \nabla p + \left(\boldsymbol{U} \cdot \nabla\right)^2 q = -\frac{h_0}{\rho_0} \frac{\partial p_n}{\partial t} - \frac{R_0}{\rho_0} p_n - \sum_{m=1}^{M} A_m p_m^{(0)} - \sum_{\ell=1}^{L} \left[B_\ell p_\ell^{(1)} + C_\ell p_\ell^{(2)} \right]$$

$$\frac{dp_m^{(0)}}{dt} + \lambda_m p_m^{(0)} = \frac{1}{\rho_0} p_n, \quad m = 1, ..., M$$

$$\frac{dp_\ell^{(1)}}{dt} + \alpha_\ell p_\ell^{(1)} + \beta_\ell p_\ell^{(2)} = \frac{1}{\rho_0} p_n, \quad \frac{dp_\ell^{(2)}}{dt} + \alpha_\ell p_\ell^{(2)} - \beta_\ell p_\ell^{(1)} = 0, \quad \ell = 1, ..., L$$

$$\frac{\partial q}{\partial t} = p$$

Time domain impedance condition between p and p_n

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$$\frac{\partial q}{\partial t} = p$$

Problem: Ingard-Myers condition can trigger Kelvin-Helmholtz instability wave due to the use of vortex sheet model

A stabilization of Ingard-Myers Impedance Condition

Truncated Ingard-Myers Impedance Boundary Condition (TIMIBC):

$$\frac{\partial p}{\partial t} + 2\boldsymbol{U} \cdot \nabla p + \cancel{(\boldsymbol{U} \cdot \nabla)^2 q} = -\frac{h_0}{\rho_0}\frac{\partial p_n}{\partial t} - \frac{R_0}{\rho_0}p_n - \sum_{m=1}^{M} A_m p_m^{(0)} - \sum_{\ell=1}^{L}\left[B_\ell p_\ell^{(1)} + C_\ell p_\ell^{(2)}\right]$$

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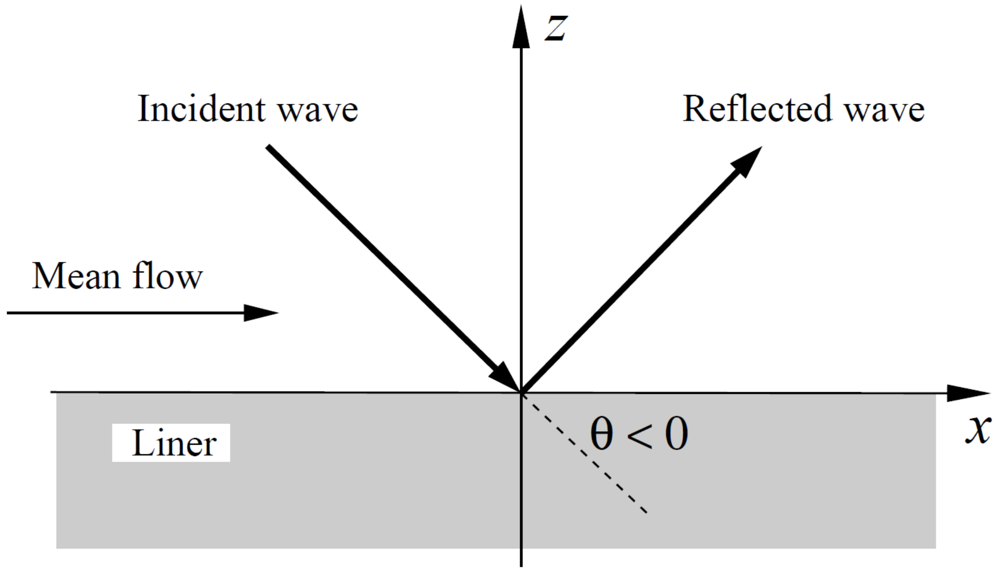
It turns out:

1) **Instability wave is eliminated (proven analytically)**
2) **Limited impact on the impedance at low to mid Mach numbers**



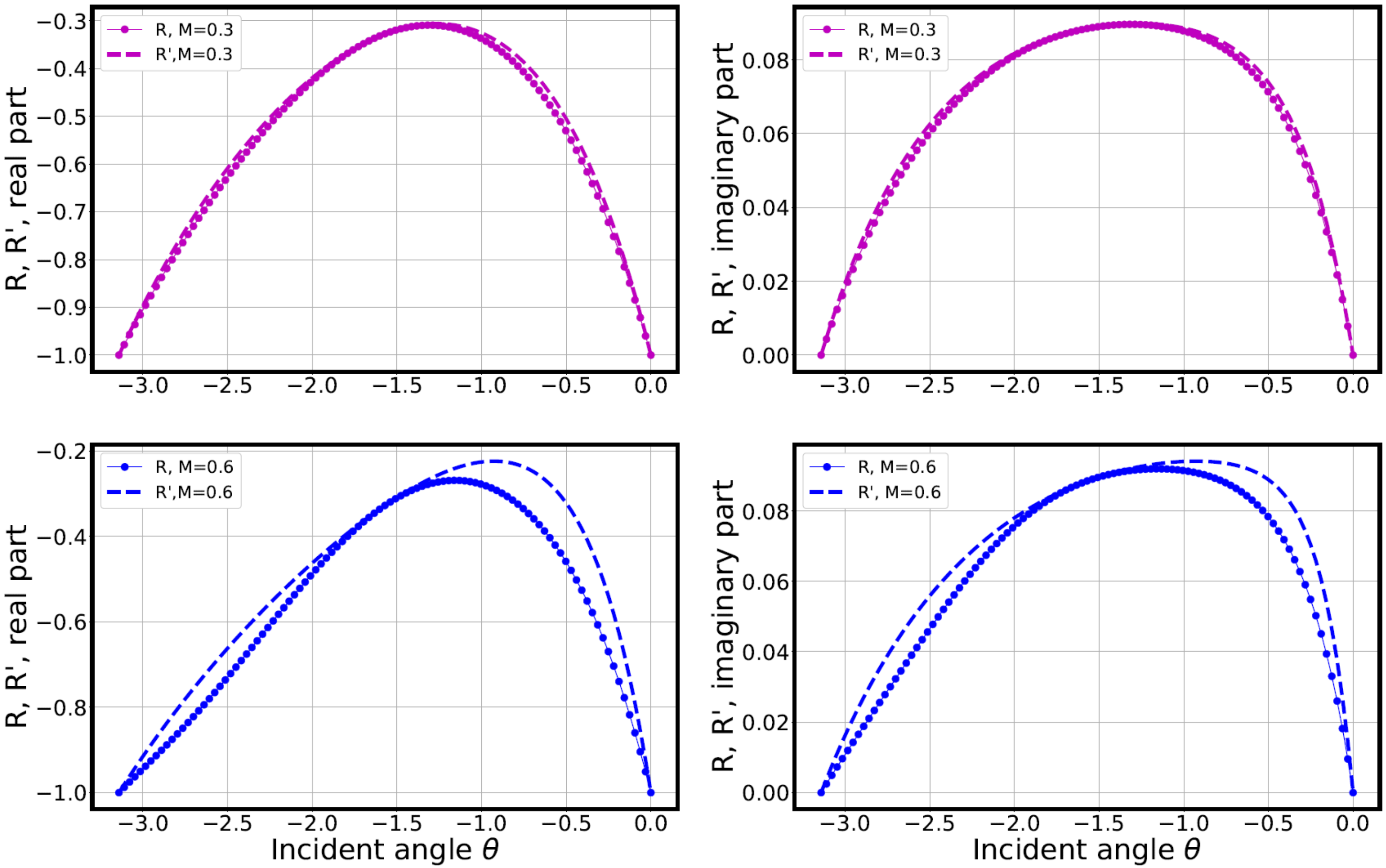
Reflection Coefficient by the Ingard-Myers Condition:

$$R = \frac{Z\,(1 + M\cos\theta)\sin\theta + \rho_0 c}{Z\,(1 + M\cos\theta)\sin\theta - \rho_0 c}$$

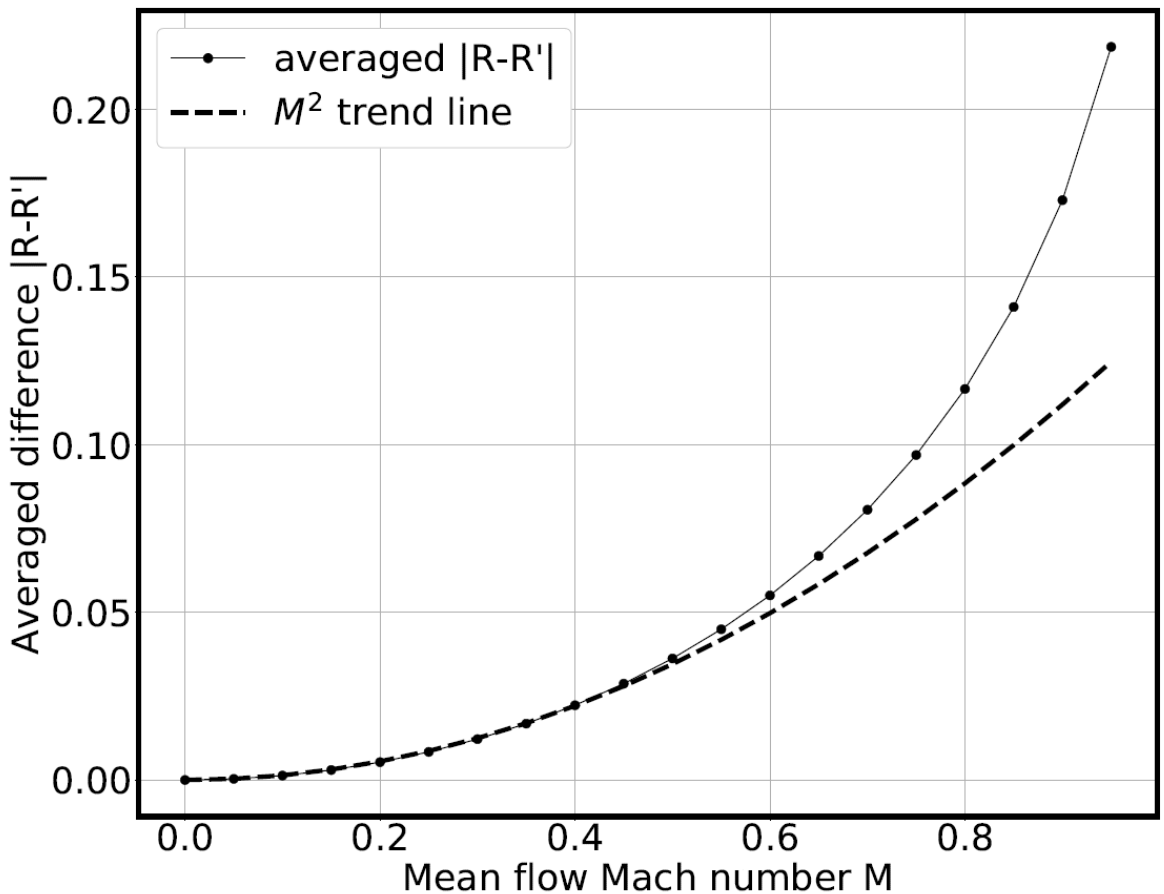
Reflection Coefficient by the Truncated Condition:

$$R' = \frac{Z\,(1 + M\cos\theta)\sin\theta + \rho_0 c(1 - M^2\cos^2\theta)}{Z\,(1 + M\cos\theta)\sin\theta - \rho_0 c(1 - M^2\cos^2\theta)}$$

A comparison of reflection coefficients



Effect of Mach number, averaged over all angles



Example: Point source reflection by lined surface

With Ingard-Myers condition, M=0.3:

Flat Plate Dimension:
2L x 2L x 0.1L

Computed by TI

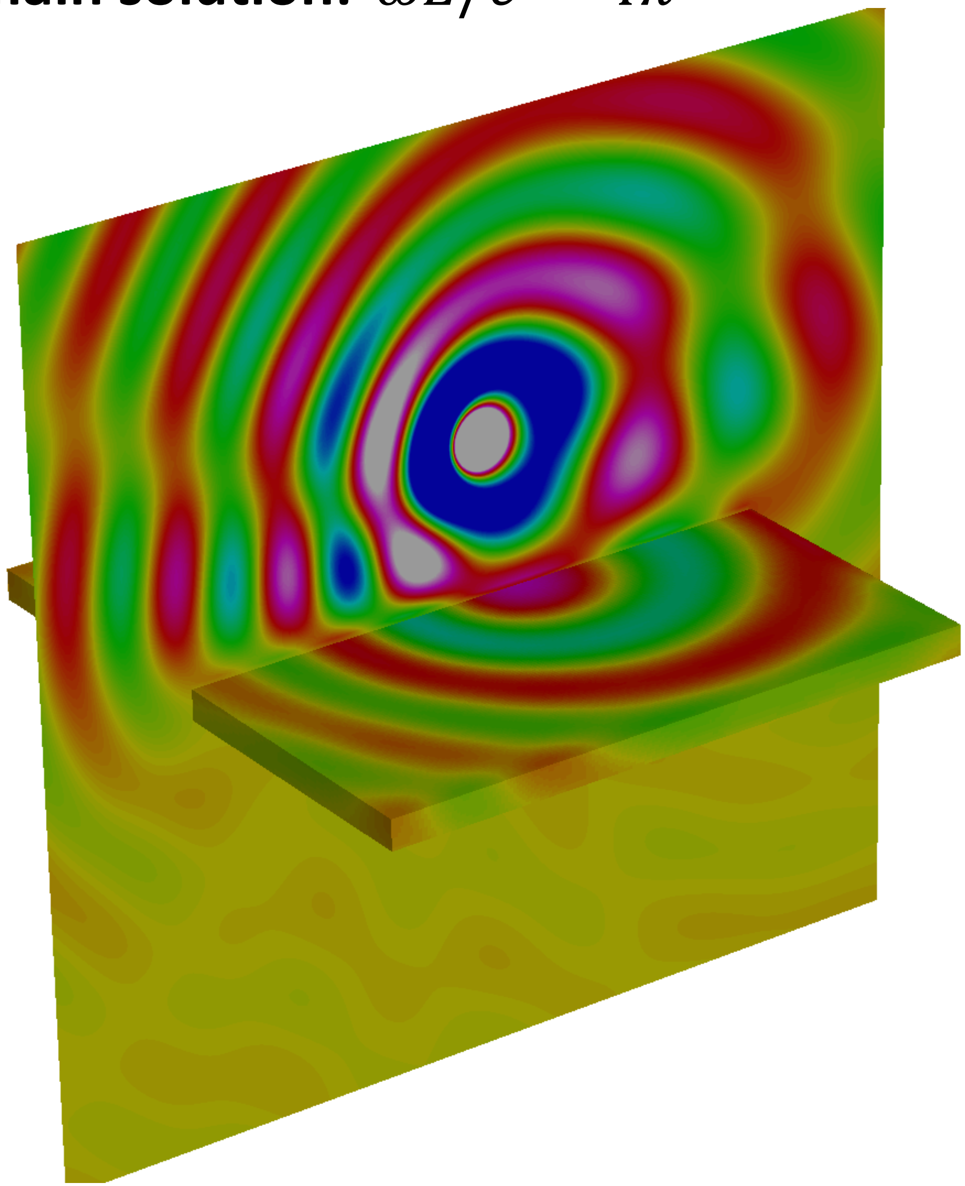
Example: Point source reflection by lined surface

With Truncated Ingard-Myers condition, M=0.3:

Flat Plate Dimension:
2L x 2L x 0.1L

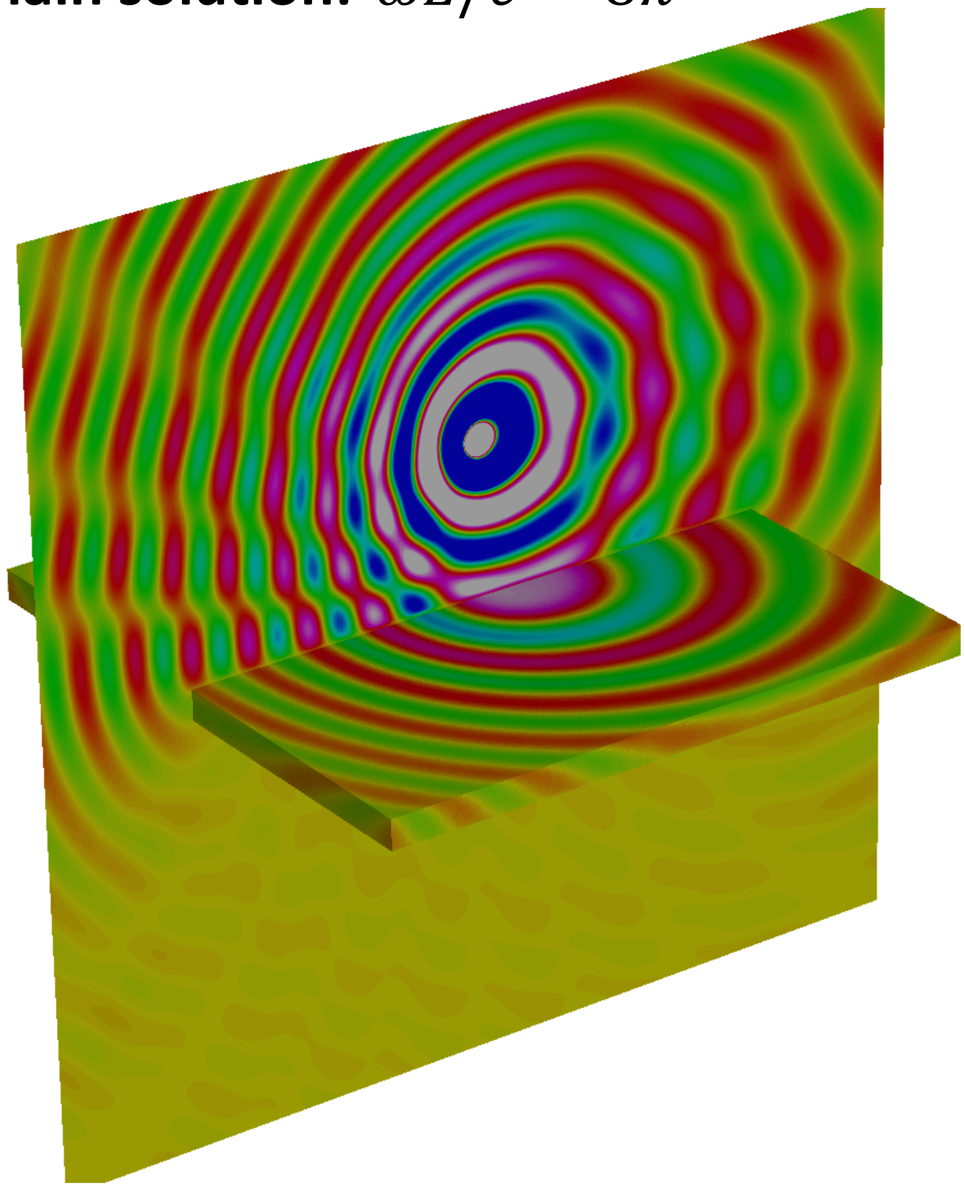
Computed by TI

Frequency domain solution: $\omega L/c = 4\pi$

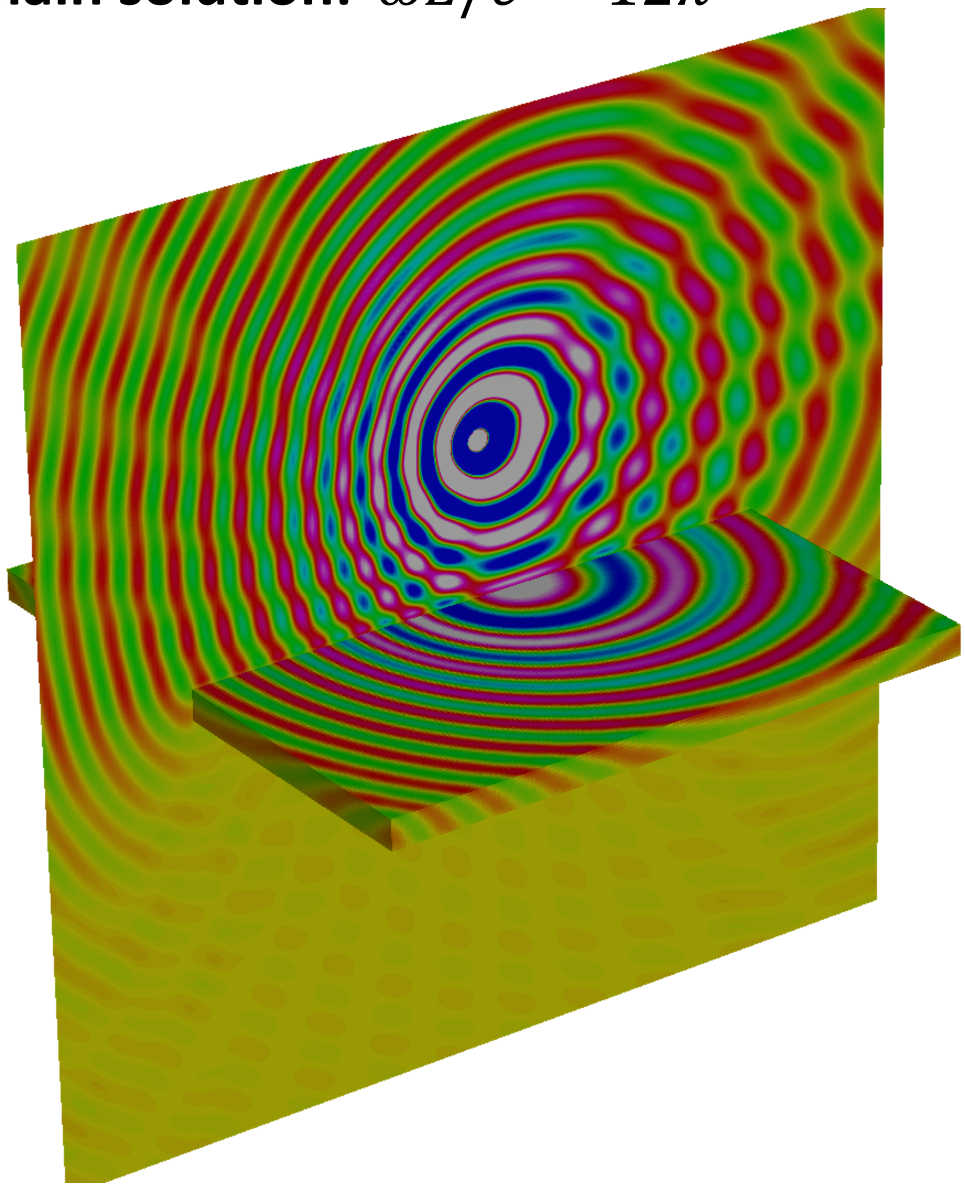


(2L: side length of plate;   c : speed of sound)

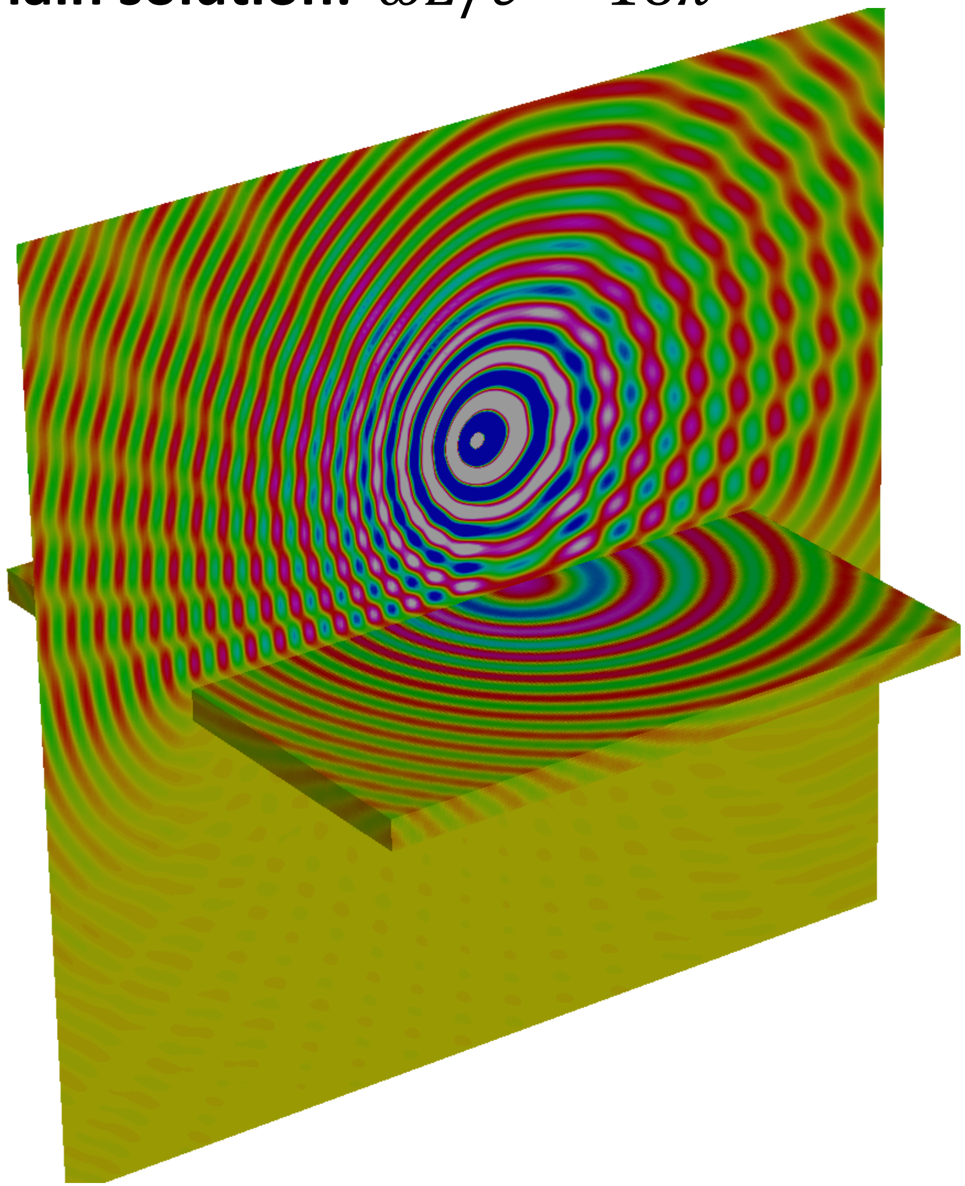
Frequency domain solution: $\omega L/c = 8\pi$



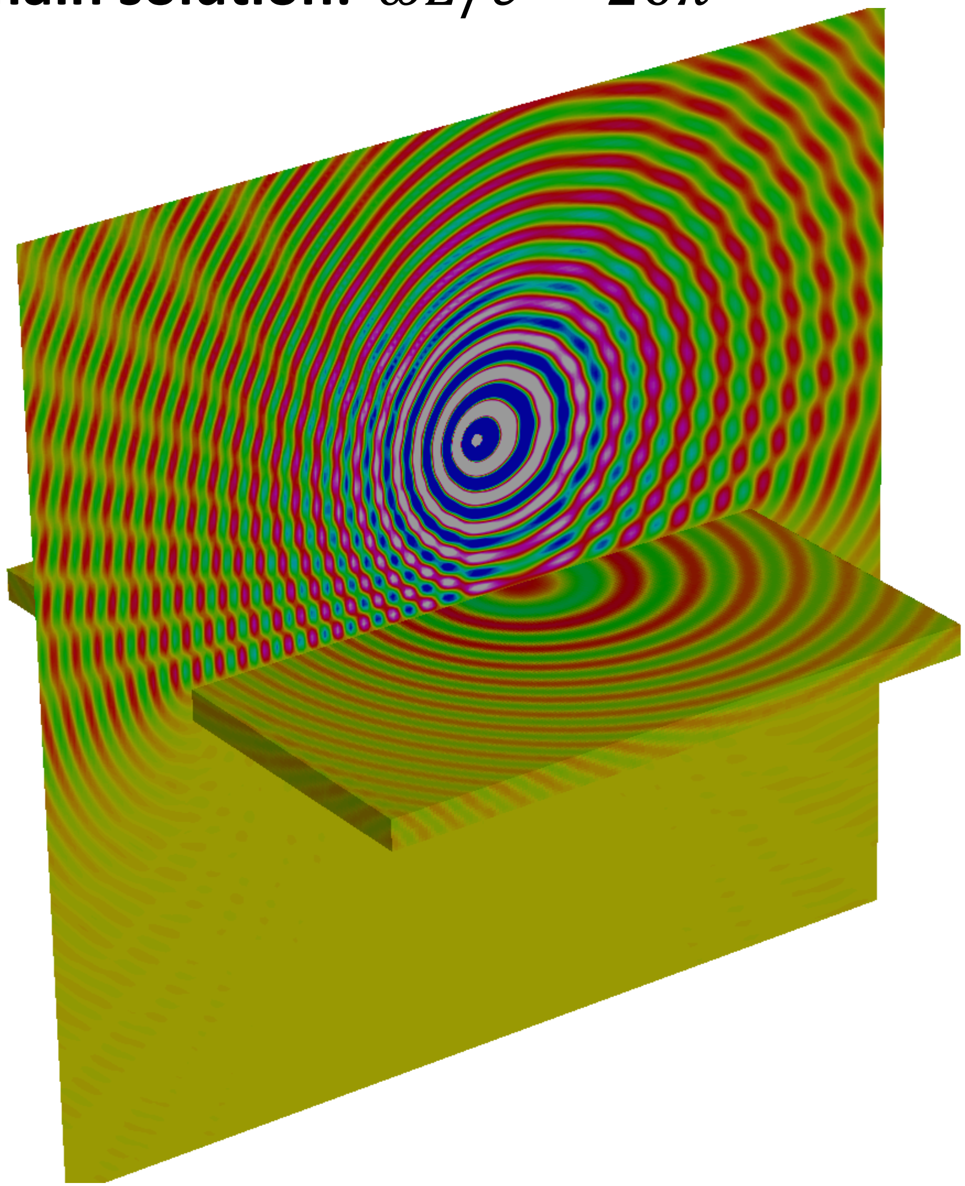
Frequency domain solution: $\omega L/c = 12\pi$



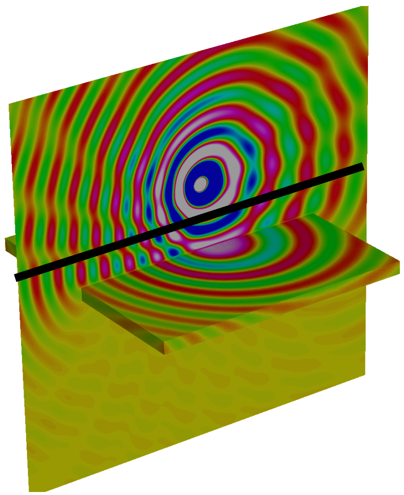
Frequency domain solution: $\omega L/c = 16\pi$

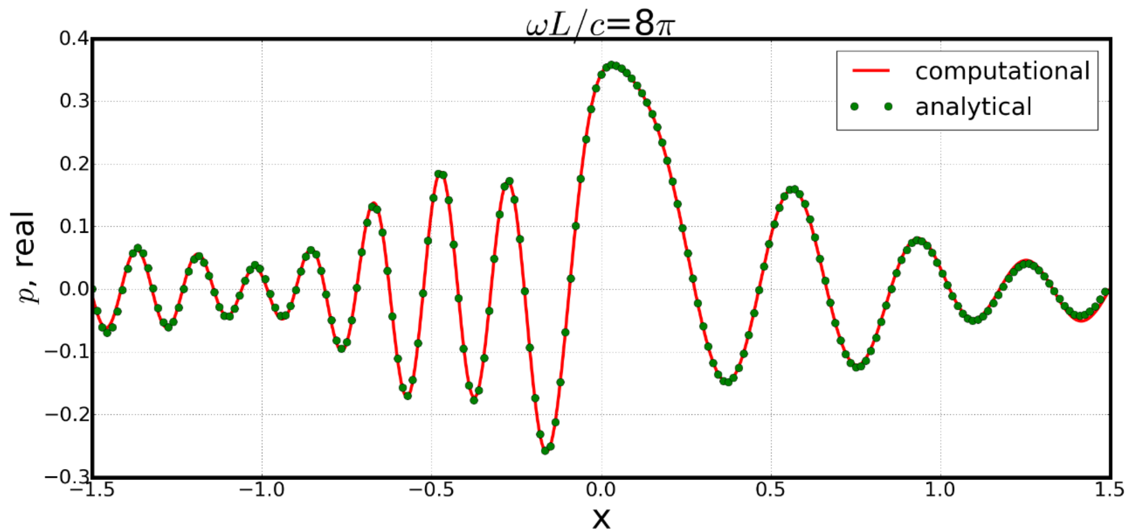


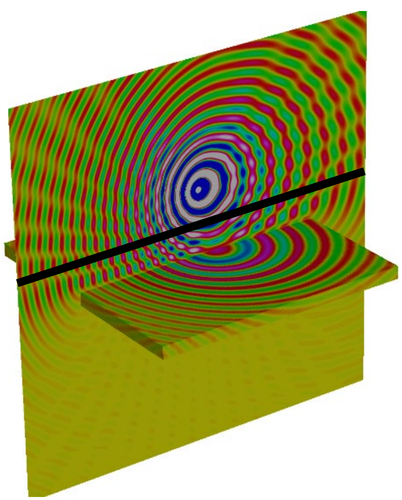
Frequency domain solution: $\omega L/c = 20\pi$

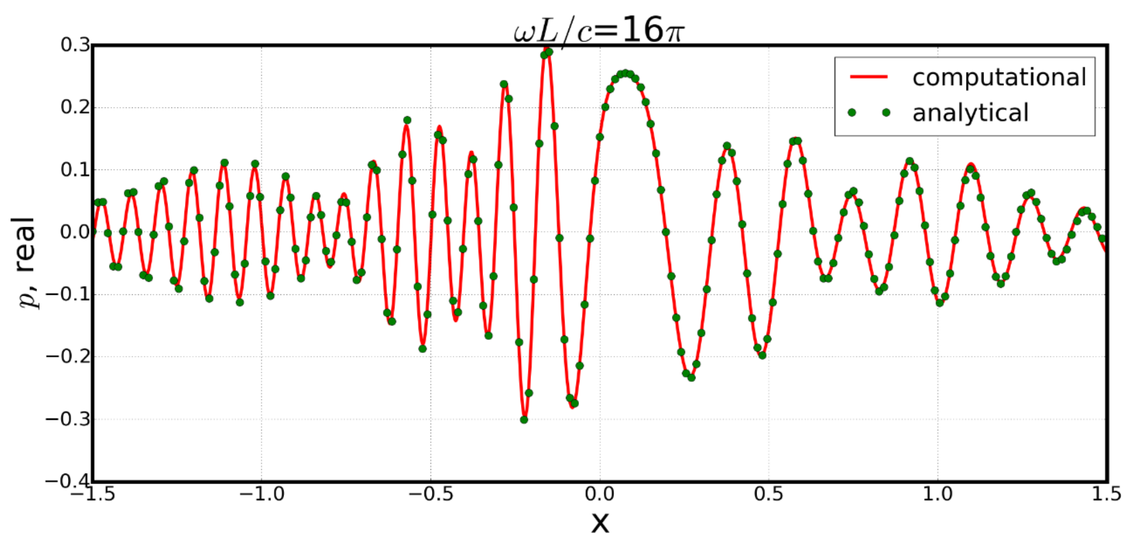


Comparisons with analytical solution

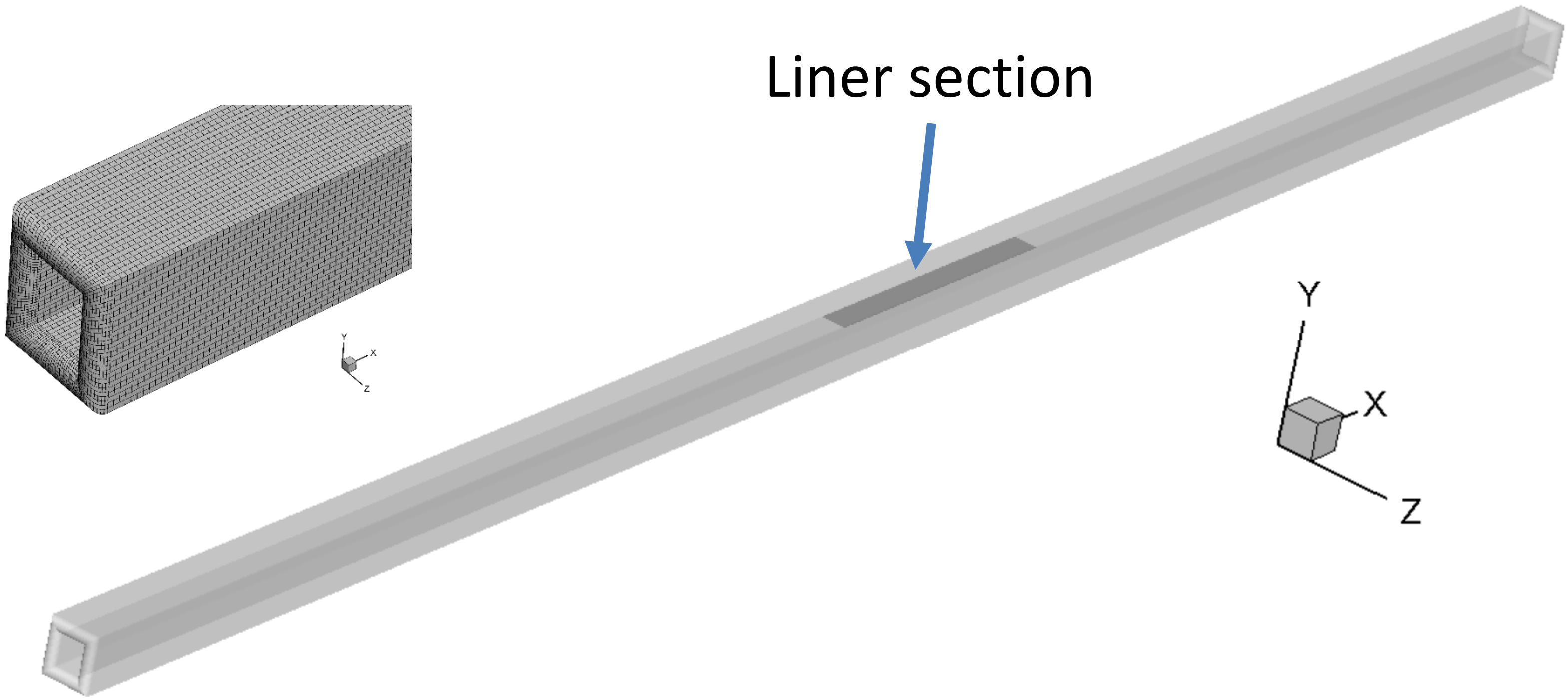








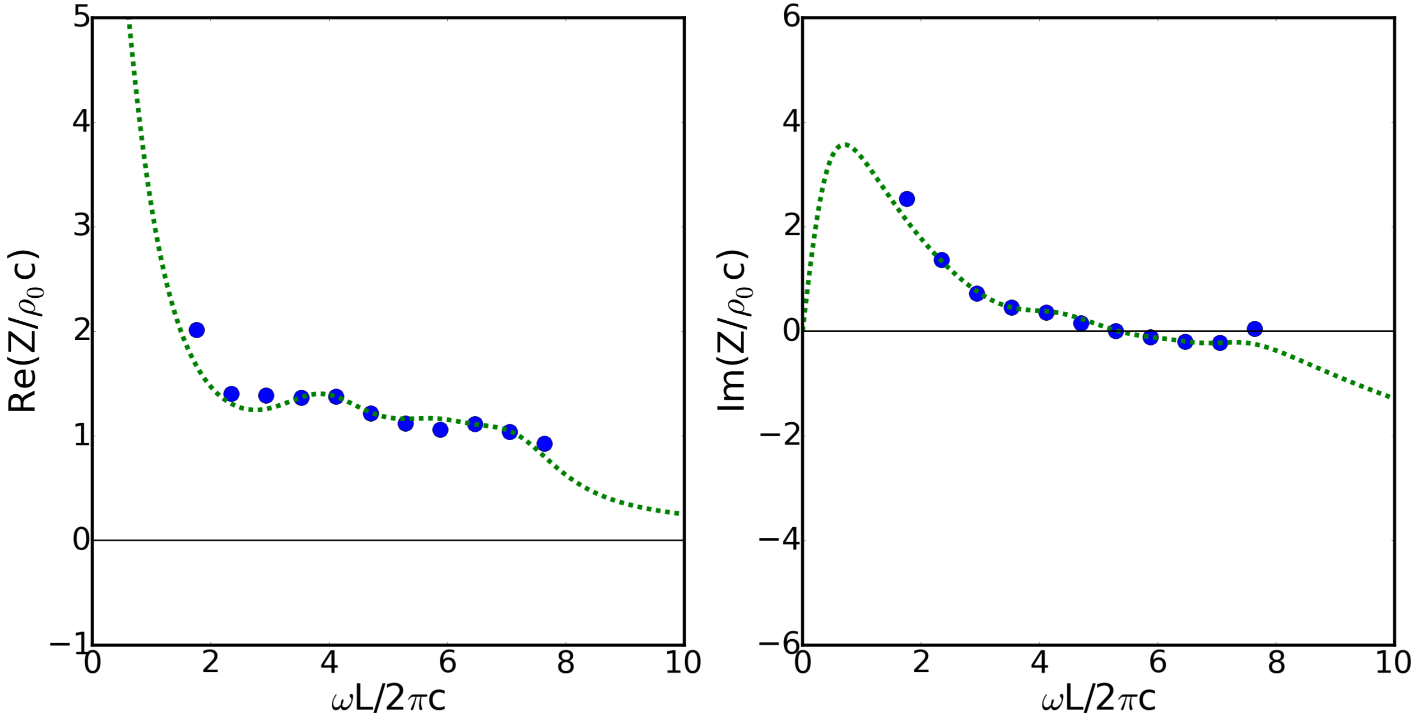
Example: Comparison with GFIT experimental dataset [1]



[1] M. Jones and W. Watson, "On the Use of Experimental Methods to Improve Confidence in Educed Impedance", AIAA paper 2011-2865, 2011.

Example of impedance function fitting

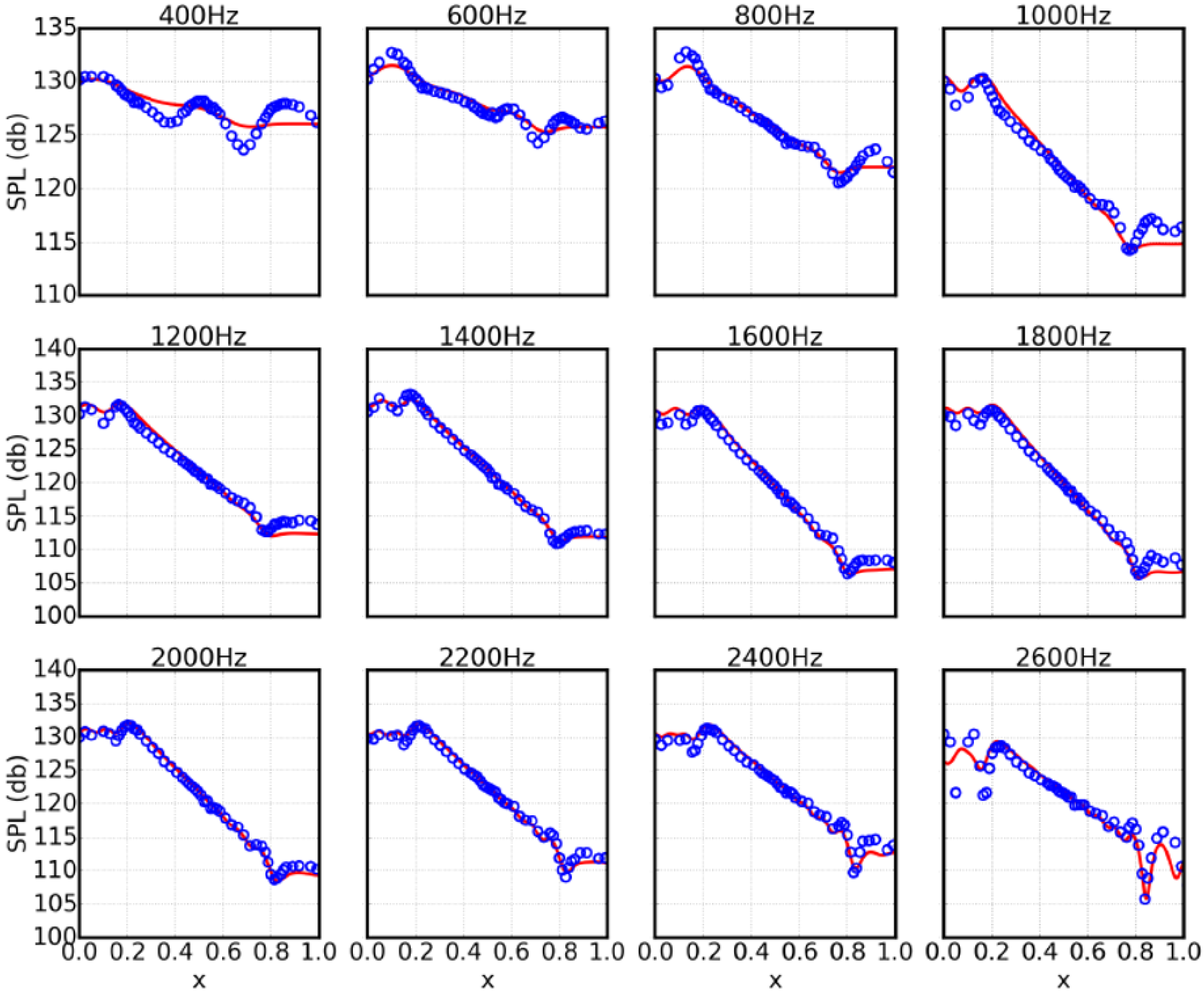
$$Z(\omega) = (-i\omega)h_0 + R_0 + \sum_{m=1}^{M} \frac{A_m}{\lambda_m - i\omega} + \frac{1}{2}\sum_{\ell=1}^{L}\left[\frac{B_\ell + iC_\ell}{\alpha_\ell + i\beta_\ell - i\omega} + \frac{B_\ell - iC_\ell}{\alpha_\ell - i\beta_\ell - i\omega}\right]$$



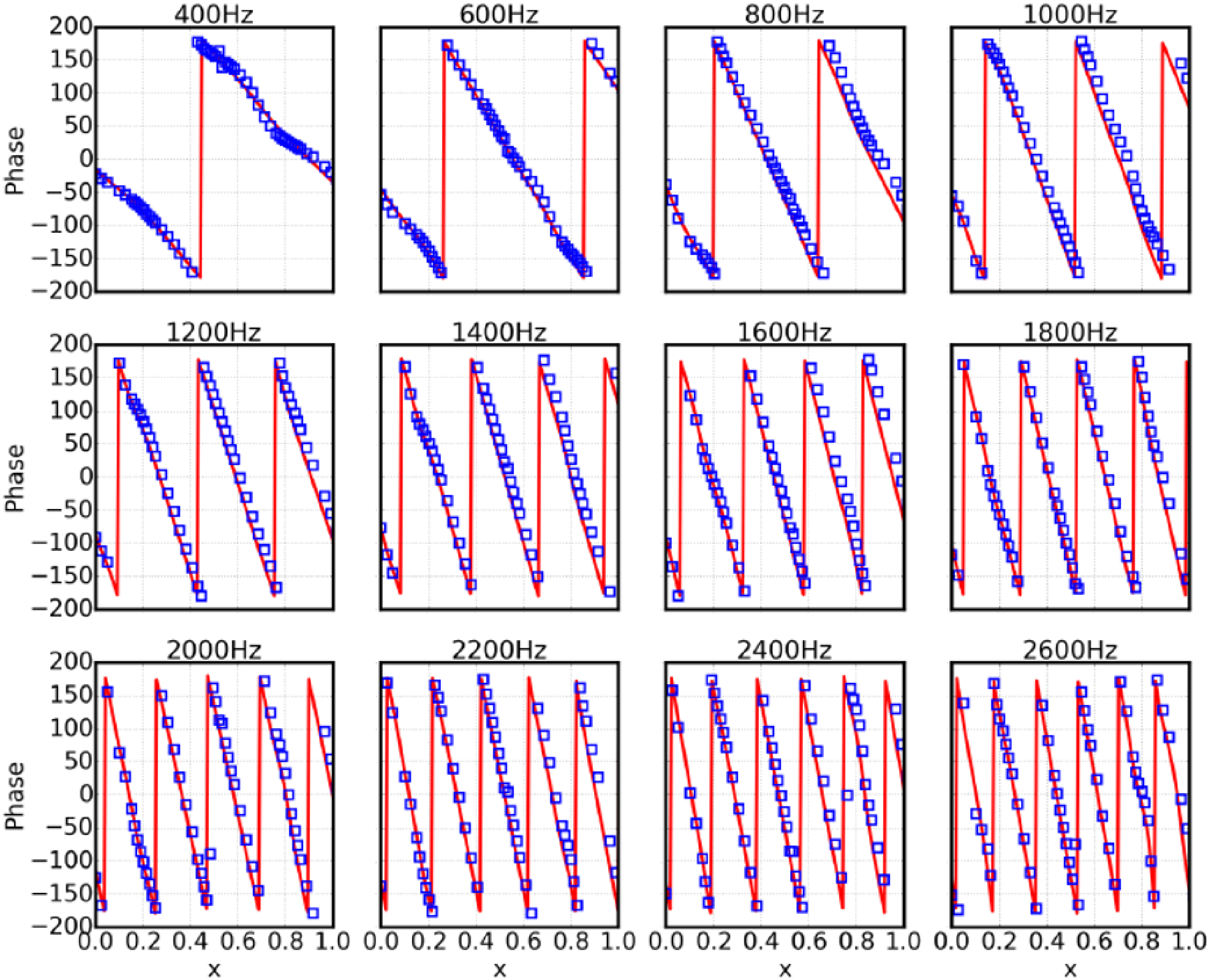
CV15R3 Liner ($M=1, L=3$)				
$h_0 = 0.039102$ $R_0 = 0.104985$	$\lambda_1 = 7.46593$ $A_1 = 6.74160$	$\alpha_1 = 7.46593$ $\beta_1 = 36.9804$	$\alpha_2 = 6.60722$ $\beta_2 = 44.9650$	$\alpha_3 = 7.69249$ $\beta_3 = 24.5147$
		$B_1 = 6.74160$	$B_2 = 7.25352$	$B_3 = 11.3198$
		$C_1 = 1.33256$	$C_2 = 0.855916$	$C_3 = 0.516327$

Wire-mesh face-sheet liner, M=0.2

Comparison with GFIT measurements, SPL:



Comparison with GFIT measurements, Phase:



Summary

- A strategy for stabilization of the Ingard-Myers impedance boundary condition has been proposed with numerical examples

- The Truncated Ingard-Myers condition has been found to be stable for a subsonic Mach number and showed to be a good approximation to the original Ingard-Myers condition at low to mid Mach numbers

- Proposed truncation can be a practical solution for avoiding the instability waves associated with the original formulation in time domain which many studies have shown to be the correct limit of boundary layer thickness going to zero

Thank you!

Backup slides

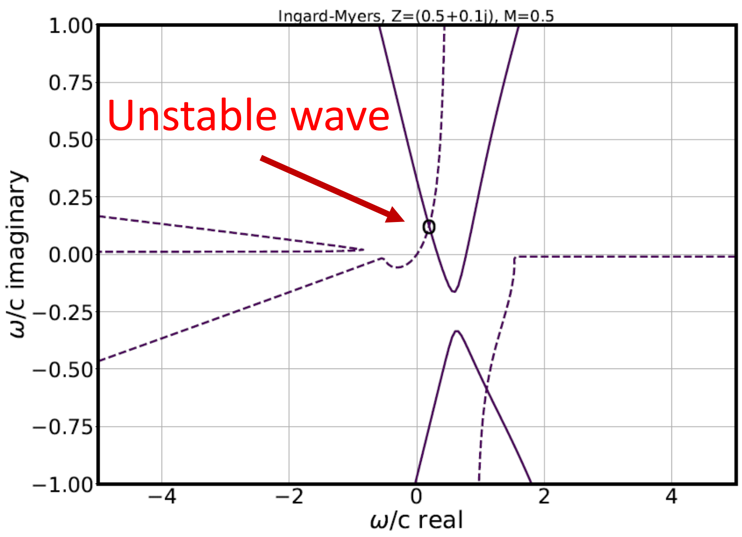
Dispersion relation for plane waves of the form $\hat{p}(\boldsymbol{r},\omega) = Ae^{ik_x x + ik_y y + i\gamma z - i\omega t}$

For the Ingard-Myers Condition:

$$\rho_0 \left(-i\omega + U\frac{\partial}{\partial x}\right)^2 \hat{p}(\boldsymbol{r},\omega) = i\omega Z(\omega)\frac{\partial \hat{p}}{\partial n}(\boldsymbol{r},\omega)$$

Dispersion relation is:

$$\boxed{(\omega/c)\gamma Z + \rho_0 c\,(\omega/c - Mk_x)^2 = 0}$$



(a)

Dispersion relation for plane waves of the form $\hat{p}(\boldsymbol{r},\omega) = Ae^{ik_x x+ik_y y+i\gamma z-i\omega t}$

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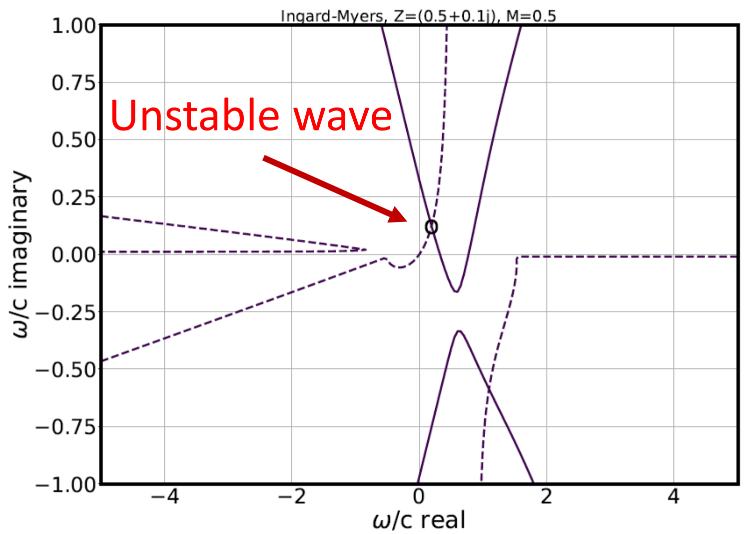


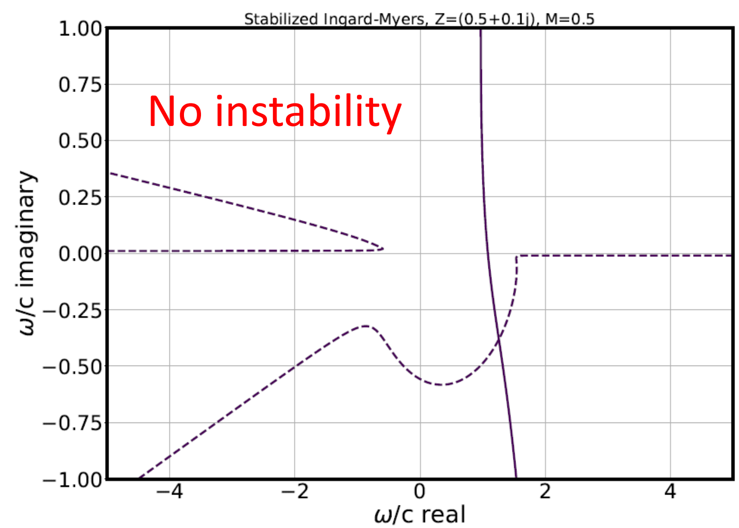
(a)

For the Truncated Ingard-Myers Condition:

$$-\rho_0\left((-i\omega)\,\hat{p} + 2U\frac{\partial \hat{p}}{\partial x}\right) = Z(\omega)\frac{\partial \hat{p}}{\partial n}$$

Dispersion relation is:

$$\boxed{\gamma Z + \rho_0 c\,(\omega/c - 2Mk_x) = 0}$$



(b)